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Advanced Linear Algebra (MA 409)
Problem Sheet - 25

Normal and Self-Adjoint Operators

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1. Label the following statements as true or false. Assume that the underlying inner product spaces are finite-dimensional.
   (a) Every self-adjoint operator is normal.
   (b) Operators and their adjoints have the same eigenvectors.
   (c) If T is an operator on an inner product space V, then T is normal if and only if $[T]_\beta$ is normal, where β is any ordered basis for V.
   (d) A real or complex matrix A is normal if and only if L_A is normal.
   (e) The eigenvalues of a self-adjoint operator must all be real.
   (f) The identity and zero operators are self-adjoint.
   (g) Every normal operator is diagonalizable.
   (h) Every self-adjoint operator is diagonalizable.

2. For each linear operator T on an inner product space V, determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.
   (a) $V = \mathbb{R}^2$ and T is defined by $T(a,b) = (2a-2b, -2a+5b)$.
   (b) $V = \mathbb{R}^3$ and T is defined by $T(a,b,c) = (-a+b, 5b, 4a-2b+5c)$.
   (c) $V = \mathbb{C}^2$ and T is defined by $T(a,b) = (2a+ib, a+2b)$.
   (d) $V = P_2(\mathbb{R})$ and T is defined by $T(f) = f'$, where
   $$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt.$$
   (e) $V = M_{2\times 2}(\mathbb{R})$ and T is defined by $T(A) = A^t$.
   (f) $V = M_{2\times 2}(\mathbb{R})$ and T is defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$.

3. Give an example of a linear operator T on $\mathbb{R}^2$ and an ordered basis for $\mathbb{R}^2$ that provides a counterexample to the statement : If T is an operator on an inner product space V, then T is normal if and only if $[T]_\beta$ is normal, where β is any ordered basis for V.

4. Let T and U be self-adjoint operators on an inner product space V. Prove that TU is self-adjoint if and only if $TU = UT$.

5. Let V be an inner product space, and let T be a normal operator on V. Prove that $T - cI$ is normal for every $c \in F$.

6. Let V be a complex inner product space, and let T be a linear operator on V. Define

$$T_1 = \frac{1}{2}(T + T^*) \quad \text{and} \quad T_2 = \frac{1}{2i}(T - T^*).$$

   (a) Prove that T_1 and T_2 are self-adjoint and that $T = T_1 + iT_2$.
   (b) Suppose also that $T = U_1 + iU_2$, where U_1 and U_2 are self-adjoint. Prove that $U_1 = T_1$ and $U_2 = T_2$.
   (c) Prove that T is normal if and only if $T_1T_2 = T_2T_1$.

7. Let T be a linear operator on an inner product space V, and let W be a T-invariant subspace of V. Prove the following results.

   (a) If T is self-adjoint, then T_w is self-adjoint.
   (b) $W^\perp$ is T^*-invariant.
   (c) If W is both T- and T^*-invariant, then $(T_W)^* = (T^*)_W$.
   (d) If W is both T- and T^*-invariant and T is normal, then T_W is normal.

8. Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a subspace of V. Prove that if W is T-invariant, then W is also T^*-invariant.

9. Let T be a normal operator on a finite-dimensional inner product space V. Prove that $N(T) = N(T^*)$ and $R(T) = R(T^*)$.

10. Let T be a self-adjoint operator on a finite-dimensional inner product space V. Prove that for all $x \in V$

$$\|T(x) \pm ix\|^2 = \|T(x)\|^2 + \|x\|^2.$$

    Deduce that $T - iI$ is invertible and that $[(T - iI)^{-1}]^* = (T + iI)^{-1}$.

11. Assume that T is a linear operator on a complex (not necessarily finite-dimensional) inner product space V with an adjoint T^*. Prove the following results.

    (a) If T is self-adjoint, then $\langle T(x), x\rangle$ is real for all $x \in V$.
    (b) If T satisfies $\langle T(x), x\rangle = 0$ for all $x \in V$, then $T = T_0$.
    Hint: Replace x by $x + y$ and then by $x + iy$, and expand the resulting inner products.
    (c) If $\langle T(x), x\rangle$ is real for all $x \in V$, then $T = T^*$.

12. Let T be a normal operator on a finite-dimensional real inner product space V whose characteristic polynomial splits. Prove that V has an orthonormal basis of eigenvectors of T. Hence prove that T is self-adjoint.

13. **Theorem :** Let T be a linear operator on a finite-dimensional real inner product space V. Then T is self-adjoint if and only if there exists an orthonormal basis β for V consisting of eigenvectors of T.

    An $n \times n$ real matrix A is said to be a **Gramian** matrix if there exists a real (square) matrix B such that $A = B^tB$. Prove that A is a Gramian matrix if and only if A is symmetric and all of its eigenvalues are nonnegative.

    Hint: Apply the above Theorem to $T = L_A$ to obtain an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of eigenvectors with the associated eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Define the linear operator U by $U(v_i) = \sqrt{\lambda_i}v_i$.

14. *Simultaneous Diagonalization*. Let V be a finite-dimensional real inner product space, and let U and T be self-adjoint linear operators on V such that $UT = TU$. Prove that there exists an orthonormal basis for V consisting of vectors that are eigenvectors of both U and T. (Note that the complex version of this result also holds good.)

    Hint: For any eigenspace $W = E_\lambda$ of T, we have that W is both T- and U-invariant and that $W^\perp$ is both T- and U-invariant.

15. Let A and B be symmetric $n \times n$ matrices such that $AB = BA$. Use Exercise 14 to prove that there exists an orthogonal matrix P such that $P^t AP$ and $P^t BP$ are both diagonal matrices.

16. Prove the Cayley Hamilton theorem for a complex $n \times n$ matrix A. That is, if $f(t)$ is the characteristic polynomial of A, prove that $f(A) = O$.

    Hint: Use Schur's theorem to show that A may be assumed to be upper triangular, in which case

$$f(t) = \prod_{i=1}^{n} (A_{ii} - t).$$

    Now if $T = L_A$, we have $(A_{jj}\mathbf{1} - T)(e_j) \in span(\{e_1, e_2, \ldots, e_{j-1}\})$ for $j \geq 2$, where $\{e_1, e_2, \ldots, e_n\}$ is the standard ordered basis for $\mathbb{C}^n$.

The following definitions are used in Exercises 17 through 23.

Definitions. A linear operator T on a finite-dimensional inner product space is called **positive definite [positive semidefinite]** if T is self-adjoint and $\langle T(x), x \rangle > 0[\langle T(x), x \rangle \geq 0]$ for all $x \neq 0$.

An $n \times n$ matrix A with entries from $\mathbb{R}$ or $\mathbb{C}$ is called **positive definite [positive semidefinite]** if L_A is positive definite [positive semidefinite].

17. Let T and U be self-adjoint linear operators on an n-dimensional inner product space V, and let $A = [T]_\beta$, where β is an orthonormal basis for V. Prove the following results.

    (a) T is positive definite [semidefinite] if and only if all of its eigenvalues are positive [non-negative].

    (b) T is positive definite if and only if

$$\sum_{i,j} A_{ij} a_j \bar{a}_i > 0 \text{ for all nonzero } n\text{-tuples } (a_1, a_2, \ldots, a_n).$$

    (c) T is positive semidefinite if and only if $A = B^* B$ for some square matrix B.

    (d) If T and U are positive semidefinite operators such that $T^2 = U^2$, then $T = U$.

    (e) If T and U are positive definite operators such that $TU = UT$, then TU is positive definite.

    (f) T is positive definite [semidefinite] if and only if A is positive definite [semidefinite].

    Because of (f), results analogous to items (a) through (d) hold for matrices as well as operators.

18. Let $T : V \to W$ be a linear transformation, where V and W are finite-dimensional inner product spaces. Prove the following results.

    (a) T^*T and TT^* are positive semidefinite.

    (b) $rank(T^*T) = rank(TT^*) = rank(T)$.

19. Let T and U be positive definite operators on an inner product space V. Prove the following results.

(a) $T + U$ is positive definite.
(b) If $c > 0$, then cT is positive definite.
(c) T^{-1} is positive definite.

20. Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle$, and let T be a positive definite linear operator on V. Prove that $\langle x, y \rangle' = \langle T(x), y \rangle$ defines another inner product on V.

21. Let V be a finite-dimensional inner product space, and let T and U be self-adjoint operators on V such that T is positive definite. Prove that both TU and UT are diagonalizable linear operators that have only real eigenvalues.

    Hint: Show that UT is self-adjoint with respect to the inner product $\langle x, y \rangle' = \langle T(x), y \rangle$. To show that TU is self-adjoint, repeat the argument with T^{-1} in place of T.

22. This exercise provides a converse to Exercise 20. Let V be a finite-dimensional inner product space with inner product $\langle \cdot, \cdot \rangle$, and let $\langle \cdot, \cdot \rangle'$ be any other inner product on V.

    (a) Prove that there exists a unique linear operator T on V such that $\langle x, y \rangle' = \langle T(x), y \rangle$ for all x and y in V.
    Hint: Let $\beta = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V with respect to $\langle \cdot, \cdot \rangle$, and define a matrix A by $A_{ij} = \langle v_j, v_i \rangle'$ for all i and j. Let T be the unique linear operator on V such that $[T]_\beta = A$.
    (b) Prove that the operator T of (a) is positive definite with respect to both inner products.

23. Let U be a diagonalizable linear operator on a finite-dimensional inner product space V such that all of the eigenvalues of U are real. Prove that there exist positive definite linear operators T_1 and T_1' and self-adjoint linear operators T_2 and T_2' such that $U = T_2 T_1 = T_1' T_2'$.

    Hint: Let $\langle \cdot, \cdot \rangle$ be the inner product associated with V, β a basis of eigenvectors for U. $\langle \cdot, \cdot \rangle'$ the inner product on V with respect to which β is orthonormal, and T_1 the positive definite operator according to Exercise 22. Show that U is self-adjoint with respect to $\langle \cdot, \cdot \rangle'$ and $U = T_1^{-1} U^* T_1$ (the adjoint is with respect to $\langle \cdot, \cdot \rangle$). Let $T_2 = T_1^{-1} U^*$.

24. This argument gives another proof of Schur's theorem. Let T be a linear operator on a finite dimensional inner product space V.

    (a) Suppose that β is an ordered basis for V such that $[T]_\beta$ is an upper triangular matrix. Let γ be the orthonormal basis for V obtained by applying the Gram Schmidt orthogonalization process to β and then normalizing the resulting vectors. Prove that $[T]_\gamma$ is an upper triangular matrix.
    (b) Recall that if the characteristic polynomial of T splits, then there is an ordered basis β for V such that $[T]_\beta$ is an upper triangular matrix.
    (c) Use (b) and (a) to obtain an alternate proof of Schur's theorem.

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